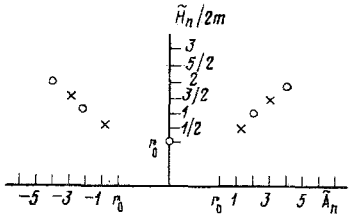

Fig. 1

either by solving the Schrödinger equation or by studying the representations of the superconformal group, which in the given case is a dynamical symmetry group. The mechanism of spontaneous breaking of the superconformal group proposed in the paper affects the time variable nontrivially, and therefore it would be interesting to look for generalizations of the proposed mechanism to more realistic models. It should also be noted that the example in the paper of construction of a dimensionless quantity by means of a limiting process indicates that one must exercise great caution with regard to arguments based on dimensional considerations in supergravity too.

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AN ALGORITHM FOR CALCULATING MULTILOOP INTEGRALS

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A class of p integrals which arise in perturbative calculations of some problems in quantum field theory is considered. An algorithm of analytic calculation of any p integral at the three-loop level convenient for realization in the existing systems of analytic calculation is constructed.

1. The renormalization group is one of the main tools of quantum field theory [1]. Therefore, the problem of calculating the coefficients of the renormalization-group equations is of fundamental importance. There are also many physically interesting problems that can be solved by the methods developed for renormalization-group calculations (see the review [2]), the characteristic feature in such cases being the inapplicability of approximate calculations [2].

It is well known that the calculation of renormalization-group quantities actually reduces to calculation of the renormalization counterterms of the corresponding Feynman diagrams. Further, as is shown in [3], the problem of calculating the counterterm for any Feynman diagram can be reduced in the framework of dimensional regularization and the minimal subtraction scheme to the calculation as far as the finite part of some massless integral of Feynman type with one external momentum and having one momentum integration less than the original diagram. The integrand consists of a polynomial in the internal momenta and the external momentum, which occurs in the numerator and the denominator and is formed by several "propagators" of the form $(q^2)^{-n}$, where n is an integer larger than zero. We shall call such integrals p integrals. We emphasize once more that the problem of renormalization-group calculations in l loops is equivalent to the calculation to the finite part of p integrals with $l - 1$ loops.

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In [4, 5] a Gegenbauer polynomial technique was developed in the coordinate space, making it possible to calculate all two-loop p integrals, and also to solve more complicated problems [6]. However, most of the problems containing calculations of three-loop p integrals are so cumbersome that they require the use of a computer for the analytic calculations. This imposes on the method of calculation the additional requirement of simplicity of programming, which is not the case for the Gegenbauer polynomial technique.

To overcome this difficulty, a recursive algorithm was constructed in [7, 8] on the basis of integration by parts; in principle, it permits the calculation of all three-loop p integrals. However, it is well known that recursive algorithms are, as a rule, much less effective on a computer than nonrecursive algorithms. In addition, the direct realization of the algorithm of [8] on the analytic-calculation system SCHOONSCHIP [9], apparently the fastest, leads to the generation of an appreciable number of similar terms in intermediate calculations, which requires excessive loss of computer time on their analysis and reduction.

A characteristic feature of the algorithm of [8] is the repeated use of one recursion relation ("triangle rule"), and also a number of auxiliary relations. In the present paper, we give an explicit recursive solution obtained by the triangle rule that makes it possible to significantly increase the output of programs for systems of the SCHOONSCHIP type by the use of built-in means for substituting the explicit expression instead of a recursive procedure. In fact, the need for the auxiliary relations then disappears as well. After this the only recursive part of the algorithm is in fact the evaluation of nonplanar integrals (see [8]), though the proportions of them in real calculations is small.

2. The basic idea of the algorithm is to apply identities of the type $\int \mathrm{div}=0$ to dimensionally-regularized integrals [7]. Simple algebraic manipulations make it possible to transform these identities into relations between different p integrals. In [7], a prescription was found (the triangle rule) for constructing such identities, these making it possible after repeated application to express planar p integrals in terms of integrals that can be calculated by successive application of the well-known formula of single-loop integration of massless integrals (see Eq. (5) below).

We demonstrate the triangle rule in a simple example. Consider the two-loop p integral

$$G(\alpha\ \beta, n_1, n_2, n_3)=\int d^Dp\, d^Dq\ (q^{2\alpha}(q+k)^{2\beta}(p-q)^{2n_2}(p+k)^{2n_1}p^{2n_3})^{-1}, \tag{1}$$

where D is the complex parameter of the dimension of space, and α and β are arbitrary and n_1, n_2, n_3 are integral positive numbers.

In the framework of dimensional regularization [10] identities of the following form hold for Feynman type integrals (including ones that diverge in the limit when the regularization is lifted):

$$\int d^Dq \frac{\partial}{\partial q} f(q)=0. \tag{2}$$

Applying to the integrand of Eq. (1) the operator

$$\frac{\partial}{\partial q_\mu}(p-q)_\mu, \tag{3}$$

and using the identity (2) and algebraic manipulations, we obtain the recursion relation

$$G(\alpha, \beta, n_1, n_2, n_3)=(D-2n_2-\alpha-\beta)^{-1}[\alpha(G(\alpha+1, \beta, n_1, n_2-1, n_3)-$$
$$G(\alpha+1, \beta, n_1-1, n_2, n_3))+\beta(G(\alpha, \beta+1, n_1, n_2-1, n_3)-G(\alpha, \beta+1, n_1, n_2, n_3-1))]. \tag{4}$$

Since the values of the integral (1) for n_1, n_2, or n_3 equal to zero can be readily found by double application of the formula (and its generalizations, see [7, 8])

$$\int d^Dp\, p^{-2\alpha}(p-k)^{-2\beta}=\pi^{D/2}(k^2)^{D/2-\alpha-\beta}\frac{\Gamma(\alpha+\beta-D/2)}{\Gamma(\alpha)\Gamma(\beta)}B(D/2-A, D/2-\beta), \tag{5}$$

it is easy to see that the recursion (4) is thus guaranteed by the boundary conditions for the calculation of (1) for admissible values of the parameters. The choice of the differential operator (3) is due to the presence of the triangle subgraph corresponding to the lines α, β, and n_3 in the graphical representation of the integral (1).

3. In the general case [8], it is necessary to solve the following recursion. Consider the

expression

$$F(n_1\ n_2, n_3, \alpha, \beta)=\Gamma(\alpha)\Gamma(\beta)\int d^D q q^{\mu_1}\ldots q^{\mu_m}((q+p_1)^{2\alpha}(q+p_2)^{2\beta}p_1^{2n_1}\ p_2\ \ q^{2n_2})^{-1} \tag{6}$$

under the same restrictions on the parameters. Using the operator $\frac{\partial}{\partial q_\mu}q^\mu$, we obtain in complete analogy with the previous section the recursion relation

$$F(n_1, n_2, n_3, \alpha, \beta)=(D+m-2n_2-\alpha-\beta)^{-1}[F(n_1, n_2-1, n_3, \alpha+1, \beta)-$$
$$F(n_1-1, n_2, n_3, \alpha+1, \beta)+F(n_1, n_2-1, n_3, \alpha, \beta+1)-F(n_1, n_2, n_3-1, \alpha, \beta+1)]. \tag{7}$$

The values of F for n_1, n_2, or n_3 equal to zero are regarded as boundary values.

We shall solve (6) as follows. We introduce the operators

$$\left\{\begin{matrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\\ \hat{e}_4\\ \hat{A}\end{matrix}\right\}F(n_1,\ n_2,\ n_3,\ \alpha,\ \beta)=\begin{cases}F(n_1,\ n_2-1,\ n_3,\ \alpha+1,\ \beta),\\ F(n_1-1,\ n_2,\ n_3,\ \alpha+1,\ \beta),\\ F(n_1,\ n_2-1,\ n_3,\ \alpha,\ \beta+1),\\ F(n_1,\ n_2,\ n_3-1,\ \alpha,\ \beta+1),\\ (D+m-2n_2-\alpha-\beta)\times\\ \times F(n_1,\ n_2,\ n_3,\ \alpha,\ \beta).\end{cases} \tag{8}$$

Then (7) can be rewritten in the form

$$F=(\hat{e}_1+\hat{e}_2+\hat{e}_3-\hat{e}_4)\hat{A}^{-1}F. \tag{9}$$

The operators (8) satisfy the obvious commutation relations

$$\hat{A}^{-1}\hat{e}_{1,3}=\hat{e}_{1,3}(\hat{A}+1)^{-1},\quad \hat{A}^{-1}\hat{e}_{2,4}=\hat{e}_{2,4}(\hat{A}-1)^{-1}. \tag{10}$$

The unique linear combination of α, β, and n_i conserved under the action of all $\hat{e}_i$ is

$$I=\alpha+\beta+n_1+n_2+n_3. \tag{11}$$

The result of applying the recursion (9) to $F(N_1, N_2, N_3, A, B)$ is a sum of the form

$$F(N_i, A, B)=\sum_{\{n_i,\alpha\}\in O}c(n_i, N_i, A, B, \alpha)F(n_i, \alpha, \beta), \tag{12}$$

where β is a uniquely determined (see (11)) function of N_i, A, B, α, and n_i, and O consists of three subregions O_j, j = 1, 2, 3:

$$O_1=\{n_i, \alpha\,|\,n_1=0,\ 0<n_2\leqslant N_2,\ 0<n_3\leqslant N_3\},\quad O_2=\{n_i, \alpha\,|\,0<n_1\leqslant N_1,\ n_2=0,\ 0<n_3\leqslant N_3\},$$
$$O_3=\{n_i, \alpha\,|\,0<n_1\leqslant N_1,\ 0<n_2\leqslant N_2,\ n_3=0\}. \tag{13}$$

For all three subregions, $\alpha - A$ is a non-negative integer. We shall denote the corresponding coefficients by c_j, j = 1, 2, 3.

Thus, it is necessary to find the coefficients c in (12). We fix a point $\{n_i, \alpha\}\in O$. Obviously, if we apply the recursion (9) without reduction of such terms, $c(n_i, N_i)$ will contain a sum of terms corresponding to all possible paths that can be taken to the point $\{n_i, \alpha\}$ from $\{N_i, A\}$ under the action of the operators $\hat{e}_i$, it being necessary that the intermediate points of the paths do not belong to the boundary of O. The contribution from each path is known, and it is necessary to find the complete sum.

For given point $\{n_i, \alpha\}\in O$ the number k_i of each of the operators $\hat{e}_i$ in an arbitrary sequence

$$\{\hat{e}_{i_1},\ldots,\hat{e}_{i_K}\} \tag{14}$$

leading from $\{N_i, A\}$ to $\{n_i, \alpha\}$ can be determined uniquely:

$$k_1=\alpha+n_1-A-N_1,\quad k_2=N_1-n_1,\quad k_3=\beta+n_3-B-N_3,\quad k_4=N_3-n_3,\quad k_1+k_3=N_2-n_2,\quad K\equiv\sum_i k_i=\sum_i(N_i-n_i). \tag{15}$$

The condition for terminating the recursion on arrival at the boundary has the form (to be specific, we consider the subregion O_3)

$$\hat{e}_{i_K}=\hat{e}_4. \tag{16}$$

We now have

$$c_j(n_i, N_i, \alpha)F(n_i, \alpha, \beta) = (-)^{k_2+k_4}\sum (\hat{e}_{i_K}\hat{A}^{-1})\ldots(\hat{e}_{i_1}\hat{A}^{-1})F(N_i, A, B), \tag{17}$$

where the sum is taken over all sets (14) satisfying (15) and (16). For c_3, we have in operator form

$$c_3(n_i, N_i, \alpha) = (-)^{k_2+k_4}\hat{e}_4\hat{A}^{-1}\sum (\hat{e}_{i_{K-1}}\hat{A}^{-1})\ldots(\hat{e}_{i_1}\hat{A}^{-1}), \tag{18}$$

whence, using (10), we obtain

$$c_3(n_i, N_i, \alpha) = (-)^{k_2+k_4}\hat{e}_4\sum (\hat{e}_{i_{K-1}}\hat{A}^{-1})\ldots(\hat{e}_{i_1}\hat{A}^{-1})(\hat{A}+k_1+k_3-k_2-k_4+1)^{-1}. \tag{19}$$

From the point of view of calculation of the coefficients c we need not distinguish $\hat{e}_1$ and $\hat{e}_3$ or $\hat{e}_2$ and $\hat{e}_4$ (see (6)), and therefore we introduce $\hat{e}_+$ instead of $\hat{e}_1$ and $\hat{e}_3$ and $\hat{e}_-$ instead of $\hat{e}_2$ and $\hat{e}_4$ in order to preserve the properties of commutation with $\hat{A}$. Then

$$c_3(n_i, N_i, \alpha) = (-)^{k_2+k_4}\hat{e}_4\binom{k_1+k_3}{k_1}\binom{k_2+k_4-1}{k_2}Y(k_1+k_3, k_2+k_4-1)(\hat{A}+k_1+k_3-k_2-k_4+1)^{-1}, \tag{20}$$

where $\binom{a}{b}$ is the binomial coefficient, and

$$Y(k_+, k_-) = \sum_{\varepsilon_K=\pm}\ldots\sum_{\varepsilon_1=\pm}(\hat{e}_{\varepsilon_K}\hat{A}^{-1})\ldots(\hat{e}_{\varepsilon_1}\hat{A}^{-1}), \tag{21}$$

in which $k = k_+ + k_-$. Obviously, Y satisfies the recursion

$$Y(k_+, k_-) = \hat{e}_+\hat{A}^{-1}Y(k_+-1, k_-) + \hat{e}_-\hat{A}^{-1}Y(k_+, k_--1) =$$
$$\hat{e}_+Y(k_+-1, k_-)(\hat{A}+k_+-k_--1)^{-1} + \hat{e}_-Y(k_+, k_--1)(\hat{A}+k_+-k_-+1)^{-1} \tag{22}$$

with the initial condition

$$Y(0, 0) = 1, \quad Y(k_+, k_-) = 0, \quad k_+, k_- < 0. \tag{23}$$

It is easy to show by induction that the solution of (22) and (23) is given by

$$Y(k_+, k_-) = \binom{k_+ + k_-}{k_+}\frac{\Gamma(\hat{A}-k_-)}{\Gamma(\hat{A}+k_++1)}(\hat{A}+k_+-k_-). \tag{24}$$

Finally, we have

$$c_3(n_i, N_i, \alpha) = (-)^{k_2+k_4}\binom{k_1+k_3}{k_1}\binom{k_2+k_4-1}{k_2}\binom{k_1+k_2+k_3+k_4-1}{k_1+k_3}\frac{\Gamma(\hat{A}-k_2-k_4+1)}{\Gamma(\hat{A}+k_1+k_3+1)}. \tag{25}$$

Similarly,

$$c_1(n_i, N_i, \alpha) = c_3(n_i, N_i, \beta), \tag{26}$$

$$c_2(n_i, N_i, \alpha) = (-)^{k_2+k_4}\binom{k_1+k_3}{k_1}\binom{k_2+k_4}{k_4}\binom{k_1+k_2+k_3+k_4-1}{k_2+k_4}\frac{\Gamma(\hat{A}-k_2-k_4)}{\Gamma(\hat{A}+k_1+k_3)} \tag{27}$$

Equations (25)-(27) together with (12) provide the required solution, it being necessary to replace $\hat{A}$ by the expression $\hat{A} = D + m - 2N_2 - A - B$, and to use (15) and (11) to determine k_i.

In particular, using (5), we can write down the explicit (but cumbersome) expression for the two-loop integral (1), this being a generalization of the result obtained in [4] by means of the Gegenbauer polynomial technique.

The algorithm described in [8] now has the following general form. I. In the numerator expansion of the polynomial with respect to the basis of scalar invariants and treatment of the nonplanar terms as in [8]. II. The equations (6), (12), and (25)-(27) of the present paper are applied to the nontrivial three-loop integrals. III. Each term of the resulting expression contains a single-loop integration, which can be performed by means of the expression (5) or its generalizations given in [7, 8]. IV. The result contains nontrivial two-loop integrals of three types. Two types (the most numerous) are special cases of the integral (1), and the results of the present paper are again applied to them; the procedure described in [8] is applied to the third type, which contains only a small number of different terms because of the high symmetry.

As an example, we mention that Gorishnii (Joint Institute for Nuclear Research) and Larin (Moscow State University), with the participation of the present author, have written two program variants for the system SCHOONSCHIP (on CDC-6500 computer) which realize, respectively, the algorithm of [8] in a pure form and the algorithm that uses the results of the present paper. For comparison, the second variant was completed four times faster than the first. In addition, the significant simplification of the program in the second variant greatly facilitated its use on account of the modular structure and the additional functional possibilities, and also made possible an additional optimization, so that the final variant works twice as fast as the first approximate variant, the gain moreover increasing with increasing complexity of the calculated integrals. But the most important thing is that, because of the simplicity of the structure, it was possible to guarantee almost complete absence of mistakes in the second program, in contrast to the first (for a program involving 3000 lines of original text this is not trivial). At the present time, this variant is being intensively used for calculations of three- and four-loop complexity (examples can be found in [11, 12]). At present, other programs with such possibilities do not exist.

We conclude with the following comments. From the point of view of computer realization it is in general meaningless to distinguish solutions in the form of a closed expression or in the form of an algorithm, since any expression is ultimately an algorithm. It is more important to distinguish algorithms on the basis of the increase in computing time as a function of the complexity of the original data (in our case, as a function of n_i; see (6)). Seen in this light, the advantages of explicit expressions of the type (12) and (25)-(27) are obvious, and this was confirmed in practice. However, even when these explicit expressions are used the region of applicability of the algorithm remains limited (for example, in calculations of deep inelastic scattering [13] a restriction must be made to not too large moments of the structure functions). This is due to the fact that the time spent on purely algebraic processing (calculation of traces, reduction of traces, etc.) increases very rapidly with increasing number and complexity of the original integrals, and a way of overcoming this difficulty cannot be seen. If it is borne in mind that already in the following perturbation order the number of different topological types of integrals exceeds the number of types analyzed in [8] for four-loop renormalization-group calculations by an order of magnitude, this alone suggests that further advance through the orders of perturbation theory (i.e., calculation of renormalization-group functions in realistic models at the level of five loops or more) is hardly conceivable without radically new methods irrespective of the size and nature of the computers.

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